Gas-kinetic derivation of Navier-Stokes-like traffic equations

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Macroscopic traffic models have recently been severely criticized as based on lax analogies only and having a number of deficiencies. Therefore, this paper shows how to construct a logically consistent fluid-dynamic traffic model from basic laws for the acceleration and interaction of vehicles. These considerations lead to the gas-kinetic traffic equation of Paveri-Fontana. Its stationary and spatially homogeneous solution implies equilibrium relations for the "fundamental diagram," the variance-density relation, and other quantities that are partly difficult to determine empirically. Paveri-Fontana's traffic equation allows the derivation of macroscopic moment equations that build a system of nonclosed equations. This system can be closed by the well proved method of Chapman and Enskog, which leads to Euler-like traffic equations in zeroth-order approximation and to Navier-Stokes-like traffic equations in first-order approximation. The latter are finally corrected for the finite space requirements of vehicles. It is shown that the resulting model is able to withstand the above mentioned criticism.

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I. INTRODUCTION

Because of analogies with gas theory [1–4] and fluid dynamics [5–9,3,10] modeling and simulating traffic flow increasingly attracts the attention of physicists [1,5,8,9,11–14]. However, due to the great importance of efficient traffic for modern industrialized countries, the investigation of traffic flow has already a long tradition. In the 1950s Lighthill and Whitham [10] as well as Richards [15] proposed a first *fluid-dynamic (macroscopic)* traffic model. During the 1960s traffic research focused on *microscopic follow-the-leader models* [16–23]. *Mesoscopic* models of a *gas-kinetic (Boltzmann-like)* type came up in the 1970s [24,25,4,3,2,26]. Since the 1980s *simulation models* [27,28] play the most important role due to the availability of cheap, fast, and powerful computers. We can distinguish *macroscopic* traffic simulation models [29–32], *microscopic* simulation models [33–36] which include *cellular automaton models* [37–39,11–14], and mixtures of both [40].

In *high-fidelity* microscopic traffic models each car is described by its own equation(s) of motion. Consequently, computer time and memory requirements of corresponding traffic simulations grow proportionally to the number N of simulated cars. Therefore, this kind of model is mainly suitable for off-line traffic simulations, detailed studies (for example, of on ramps or lane mergings), or the numerical evaluation of collective quantities [33] like the density-dependent velocity distribution, the distribution of headway distances, etc., and other quantities that are difficult to determine empirically.

For this reason, fast *low-fidelity* microsimulation models that allow bit handling have been developed for the simulation of large freeways or freeway networks [37,38]. However, although they reproduce the main effects of traffic flow, they are not very suitable for detailed *predictions* because of their coarse-grained description.

Therefore, some authors prefer macroscopic traffic models [10,41–43,30,44,5–9]. These are based on equations for collective quantities like the average spatial *density* $\rho(r,t)$ per lane (at place r and time t), the *average velocity* $V(r,t)$, and maybe also the velocity *variance* $\Theta(r,t)$. Here, simulation time and memory requirements mainly depend on the discretization Δr and Δt of space r and time t, but not on the number N of cars. Therefore, macroscopic traffic models are suitable for *real-time* traffic simulations. The quality and reliability of the simulation results mainly depend on the correctness of the applied macroscopic equations and the choice of a suitable numerical integration method. The rather old and still continuing controversy on these problems [41,45,46,42,43,47,30,44,48–50,8,9,5,1,51] shows that they are not at all trivial.

Some of the most important points of this controversy will be outlined in Sec. II. It will be shown that even the most advanced models still have some serious shortcomings. The main reason for this is that the proposed macroscopic traffic equations were founded on heuristic arguments or based on analogies with the equations for ordinary fluids. In contrast to these approaches, this paper will present a *mathematical derivation* of macroscopic traffic equations starting from the gas-kinetic traffic equation of Paveri-Fontana [2] which is very reasonable and seems to be superior to the one of Prigogine and co-workers [24,25,4]. The applied method is analogous to the derivation of the Navier-Stokes equations for ordinary fluids from the Boltzmann equation [52–55]. It is based on a Chapman-Enskog expansion [56,57] which is known from kinetic gas theory and leads to idealized, Euler-like equations in zeroth-order approximation and to Navier-Stokes-like equations in first-order approximation [58,55]. In this respect, the paper puts into effect the method suggested by Nelson [1]. A similar method was already applied to the derivation of fluid-dynamic

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equations for the motion of pedestrian crowds [59], but it assumed some unsatisfactory approximations.

The further procedure of this paper is as follows. Section II presents a short history of macroscopic traffic models and discusses the abilities and weaknesses of the different approaches. Section III introduces the Boltzmann-like model of Prigogine [4] and compares it with the one of Paveri-Fontana [2]. From their gas-kinetic equations macroscopic ("fluid-dynamic") traffic equations will be derived in Sec. IV. Unfortunately, they turn out to build a hierarchy of nonclosed equations, i.e., the density equation depends on average velocity V and the velocity equation on velocity variance Θ, etc. Therefore, a suitable approximation must be found to obtain a set of closed equations. It will be shown that some of the traffic models introduced in Sec. II correspond to zeroth-order approximations of different kinds. These, however, are not very well justified. A similar situation holds for the Euler-like traffic equations which, apart from a complementary covariance equation, contain additional terms compared with the Euler equations of ordinary fluids [58]. These stem, on the one hand, from a relaxation term which describes the drivers' acceleration towards their desired velocities. On the other hand, they originate from interactions which are connected with deceleration processes since these do not satisfy momentum and energy conservation in contrast to atomic collisions.

A very realistic, first-order approximation which is, in a certain sense, self-consistent can be found by solving the reduced Paveri-Fontana equation which is obtained from the original one by integration with respect to the desired velocity. We will utilize the fact that, according to empirical traffic data [60,61,3,62,33], the equilibrium velocity distribution has a Gaussian form. This allows the derivation of mathematical expressions for the equilibrium velocity-density relation, the "fundamental diagram" of traffic flow, and the equilibrium variance-density relation (cf. Sec. IV C). Afterwards an approximate time-dependent solution of Paveri-Fontana's equation will be calculated by use of the Euler-like equations. Due to the additional terms in Paveri-Fontana's equation compared with the Boltzmann equation the corresponding mathematical procedure is more complicated than the Chapman-Enskog expansion for ordinary gases (cf. Sec. V).

Nevertheless, it is still possible to derive correction terms of the Euler-like macroscopic traffic equations (cf. Sec. VI). These have the meaning of *transport terms* (like, e.g., the flux density of velocity variance) and are related to the finite skewness γ of the velocity distribution in nonequilibrium situations. The resulting equations are Navier-Stokes-like traffic equations which, in comparison with the ordinary Navier-Stokes equations [58], contain additional terms arising from the acceleration and interaction of vehicles. Additionally, they are complemented by a covariance equation which takes into account the tendency of drivers to adapt to their desired velocities.

Because of the one-dimensionality of the Navier-Stokes-like traffic equations no *shear viscosity* term occurs. However, in Sec. VII it is indicated how transitions between different driving modes can cause a *bulk viscosity* term. Furthermore, corrections due to finite space requirements of each vehicle (vehicle length plus safe distance) are introduced.

The resulting model overcomes the shortcomings of the former macroscopic traffic models (that are mentioned in Sec. II). Section VIII summarizes the results of the paper and gives a short outlook.

II. SHORT HISTORY OF MACROSCOPIC TRAFFIC MODELS

In 1955 Lighthill and Whitham [10] proposed the first macroscopic (fluid-dynamic) traffic model. This is based on the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho V)}{\partial r} = 0 \qquad (1)$$

which reflects conservation of the number of vehicles. For the average velocity V, Lighthill and Whitham assumed a static velocity-density relation:

$$V(r,t) := V_e[\rho(r,t)] \,. \qquad (2)$$

Inserting (2) into (1) we obtain

$$\frac{\partial \rho}{\partial t} + \left[V_e + \rho \frac{\partial V_e}{\partial \rho} \right] \frac{\partial \rho}{\partial r} = 0 \,. \qquad (3)$$

Equation (3) describes the propagation of nonlinear *"kinematic waves"* with velocity $c(\rho) = V_e(\rho) + \rho\, \partial V_e/\partial \rho$ [10,63]. In the course of time the waves develop a *shock structure*, i.e., their back becomes steeper and steeper until it becomes perpendicular, leading to discontinuous wave profiles [10,15,63].

In reality, density changes are not so extreme. Therefore, it was suggested to add a diffusion term $D\partial^2\rho/\partial r^2$ which smooths out the shock structures somewhat [63,64]. The resulting equation reads

$$\frac{\partial \rho}{\partial t} + V_e \frac{\partial \rho}{\partial r} = -\rho \frac{\partial V_e}{\partial \rho} \frac{\partial \rho}{\partial r} + D \frac{\partial^2 \rho}{\partial r^2} \,. \qquad (4)$$

For the case of a linear velocity-density relation [65]

$$V_e(\rho) := V_{\max} \left(1 - \frac{\rho}{\rho_{\max}} \right) \qquad (5)$$

it can be transformed into the Burgers equation [66]

$$\frac{\partial g}{\partial t} + g \frac{\partial g}{\partial r} = D \frac{\partial^2 g}{\partial r^2} \qquad (6)$$

which is analytically solvable [63]. Here, we have introduced the function

$$g[\rho(r,t)] := V_{\max} \left(1 - \frac{2\rho(r,t)}{\rho_{\max}} \right) \,. \qquad (7)$$

The most important restriction of models (1) and (2) as well as (4) and (2) is relation (2) which assumes that the average speed $V(r,t)$ is always in equilibrium with density $\rho(r,t)$. Therefore, these models are not suitable for the description of nonequilibrium situations occurring at on ramps, changes of the number of lanes, or stop-and-go traffic.

Consequently, it was suggested to replace relation (2) by a dynamic equation for the average velocity V. In 1971, Payne [41] introduced the *velocity equation*

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial r} = -\frac{C(\rho)}{\rho}\frac{\partial \rho}{\partial r} + \frac{1}{\tau}[V_e(\rho) - V] \tag{8a}$$

with

$$C(\rho) := -\frac{1}{2\tau}\frac{\partial V_e}{\partial \rho} = \frac{1}{2\tau}\left|\frac{\partial V_e}{\partial \rho}\right|, \tag{8b}$$

which he motivated by a heuristic derivation from a microscopic follow-the-leader model [67]. Here, $V\partial V/\partial r$ is called the "convection term" and describes velocity changes at a place r that are caused by average vehicle motion. The "anticipation term" $-(C/\rho)\partial\rho/\partial r$ was intended to account for the drivers' awareness of the traffic conditions ahead. Finally, the "relaxation term" $[V_e(\rho) - V]/\tau$ delineates an (exponential) adaptation of average velocity V to the *equilibrium velocity* $V_e(\rho)$ with a *relaxation time* τ.

Unfortunately, for bottlenecks the corresponding computer simulation program FREFLO suggested by Payne [29] produces output that "does not seem to reflect what really happens even in a qualitative manner" [46]. As a consequence, several authors have suggested a considerable number of modifications of Payne's numerical integration method or of his equations [68,42,43,47,30,44,48,49,69]. A more serious weakness of Payne's equations is that their stationary and homogeneous solution is stable with respect to fluctuations over the whole density range as can be shown by a linear stability analysis [68,45,41]. Therefore, Payne's model (1) and (8) does not describe the well known self-organization of stop-and-go waves above a critical density [43,70]. This problem is removed [45] by substituting relation (8b) by

$$C(\rho) := \frac{\partial \mathcal{P}_e}{\partial \rho} \tag{9}$$

with the *equilibrium "traffic pressure"*

$$\mathcal{P}_e(\rho) := \rho\Theta_e(\rho)\,. \tag{10}$$

The modified velocity equation reads

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial r} = -\frac{1}{\rho}\frac{\partial \mathcal{P}_e}{\partial r} + \frac{1}{\tau}[V_e(\rho) - V] \tag{11a}$$

and can be derived from the gas-kinetic (Boltzmann-like) traffic models [4,3,2] (cf. Sec. IV). For $\Theta_e(\rho)$, Phillips [3,71] suggested a relation of the form

$$\Theta_e(\rho) := \Theta_0\left(1 - \frac{\rho}{\rho_{\max}}\right). \tag{11b}$$

In contrast, Kühne [72] as well as Kerner and Konhäuser [8,9] assumed, as a first approach, Θ_e to be a positive constant:

$$\Theta_e(\rho) := \Theta_0\,. \tag{12}$$

Unfortunately, Eqs. (1) and (11a) predict the formation of shock waves as Lighthill and Whitham's equation does [43,5]. For this reason, Kühne [43,70] suggested adding a small viscosity term $\nu\partial^2 V/\partial r^2$ which smooths out sudden density and velocity changes somewhat. Then the velocity equation

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial r} = -\frac{\Theta_0}{\rho}\frac{\partial \rho}{\partial r} + \nu\frac{\partial^2 V}{\partial r^2} + \frac{1}{\tau}[V_e(\rho) - V] \tag{13}$$

results. A linear stability analysis of Kühne's equations (1) and (13) shows that these predict the self-organization of stop-and-go waves or of so-called *"phantom traffic jams"* (i.e., unstable traffic) on the condition

$$\rho_e\left|\frac{\partial V_e}{\partial \rho}\right| > \sqrt{\Theta_0}(1 + \tau\nu k^2), \tag{14}$$

where k denotes the wave number of the perturbation [73,5]. This condition is fulfilled if the equilibrium density ρ_e corresponding to the stationary and spatially homogeneous solution exceeds a *critical density* $\rho_{\rm cr}$ that depends on the concrete form of $V_e(\rho)$.

For reasons of compatibility with the Navier-Stokes equations for ordinary fluids Kerner and Konhäuser replaced Kühne's constant ν by the density-dependent relation

$$\nu(\rho) = \frac{\nu_0}{\rho} \tag{15}$$

with the constant *viscosity coefficient* ν_0. Computer simulations of their equations (1), (13), and (15) show the development of *density clusters* [8,9] if the critical density $\rho_{\rm cr}$ given by (14) and (15) is exceeded. On the basis of a very comprehensive study of cluster-formation phenomena, Kerner and Konhäuser [9] presented a detailed interpretation of stop-and-go traffic.

Despite the considerable variety of proposed macroscopic traffic models, even the most advanced of them have still some shortcomings. For example, for a certain set of parameters the mentioned models predict traffic densities that exceed the maximum admissible density $\rho_{bb} = 1/l_0$ which is the *bumper-to-bumper density* (l_0 = average vehicle length) [5]. Furthermore, in certain situations even negative velocities may occur [51]. To illustrate this, imagine a queue of vehicles of constant density ρ_0. Assume that, e.g., due to an accident that blocks the road, this queue has come to rest (i.e., $V = 0$) and that it ends at $r = r_0$ which will imply $\rho(r,t) = 0$ for $r < r_0$. Then $\partial\rho/\partial r$ diverges at a place r_0 (or is at least very large) and Eqs. (8), (11), and (13) all predict $\partial V(r_0,t)/\partial t < 0$ if $\Theta \neq 0$.

Of course, we wish to have a model that is valid not only in standard situations, but also in extreme ones. Moreover, the model should provide reasonable results not only for certain parameter values. This is particularly important for the reason that technical measures like automatic distance control may change some parameter values considerably. Nobody knows if the existing phenomenological models are still applicable then. Therefore we will derive the specific structure of the traffic model from basic principles regarding the behavior of the single driver-vehicle units and their interactions.

III. GAS-KINETIC (BOLTZMANN-LIKE) TRAFFIC MODELS

Let us assume that the motion of an individual vehicle α can be described by several variables like its place $r_\alpha(t)$, its velocity $v_\alpha(t)$, and maybe other quantities which characterize the vehicle type or driving style (the driver's personality). We can combine these quantities in a vector

$$\vec{x}_\alpha(t) := (r_\alpha(t), v_\alpha(t), \dots) \tag{16}$$

that denotes the *state* of vehicle α at a given time t. The time-dependent *phase-space density*

$$\hat{\rho}(\vec{x}, t) \equiv \hat{\rho}(r, v, \dots, t) \tag{17}$$

is then determined by the mean number $\Delta n(r, v, \dots, t')$ of vehicles that are at a place between $r - \Delta r/2$ and $r + \Delta r/2$, driving with a velocity between $v - \Delta v/2$ and $v + \Delta v/2$, ... at a time $t' \in [t - \Delta t/2, t + \Delta t/2]$:

$$\hat{\rho}(r, v, \dots, t)\, \Delta r\, \Delta v \cdots := \frac{1}{\Delta t} \int_{t-\Delta t/2}^{t+\Delta t/2} dt'\, \Delta n(r, v, \dots, t')\,. \tag{18}$$

For vehicles, the phase-space densitiy $\hat{\rho}$ is a very small quantity. Therefore, in the limit $\Delta r \to 0$, $\Delta v \to 0$, ..., $\Delta t \to 0$ it is only meaningful in the sense of the expected value of an ensemble of macroscopically identical systems [1]. The interpretation of $\hat{\rho}$ as a quantity which can describe single traffic situations is only possible for "coarse-grained averaging" where Δr, Δv, ..., and Δt must be chosen "microscopically large but macroscopically small" [1,59] or, more exactly, (1) smaller than the scale on which variations of the corresponding macroscopic quantities occur, and (2) so large that $\Delta n(r, v, \dots, t) \gg 1$ which is not always compatible with the first condition. However, in any case a suitable gas-kinetic equation for the phase-space density $\hat{\rho}$ allows the derivation of meaningful equations for collective ("macroscopic") quantities like the spatial density $\rho(r, t)$ per lane, the average velocity $V(r, t)$, and the velocity variance $\Theta(r, t)$. To obtain an equation of this kind, we will bring in the well known fact that the temporal evolution of phase-space density $\hat{\rho}$ is given by the continuity equation [74]

$$\frac{\partial \hat{\rho}}{\partial t} + \nabla_{\vec{x}} \left(\hat{\rho} \frac{d\vec{x}}{dt} \right) = \left(\frac{\partial \hat{\rho}}{\partial t} \right)_{\rm tr} \tag{19}$$

which again describes conservation of the number of vehicles, but this time in the phase space $\Omega = \{\text{all admissible states } \vec{x}\}$. Whereas $\nabla_{\vec{x}} (\hat{\rho} d\vec{x}/dt)$ reflects changes of phase-space density $\hat{\rho}$ due to a motion in *phase space* Ω with velocity $d\vec{x}/dt$, the term $(\partial \hat{\rho}/\partial t)_{\rm tr}$ delineates changes of $\hat{\rho}$ due to discontinuous transitions between states.

A. Prigogine's model

In Prigogine's model the state $\vec{x}$ is given by the place r and velocity $v = dr/dt$ of a vehicle. The transition term $(\partial \hat{\rho}/\partial t)_{\rm tr}$ consists of a *relaxation term* $(\partial \hat{\rho}/\partial t)_{\rm rel}$ and an *interaction term* $(\partial \hat{\rho}/\partial t)_{\rm int}$ [24,25,4]. Therefore, Eq. (19) assumes the explicit form

$$\frac{\partial \hat{\rho}}{\partial t} + \frac{\partial (\hat{\rho} v)}{\partial r} + \frac{\partial}{\partial v} \left(\hat{\rho} \frac{dv}{dt} \right) = \left(\frac{\partial \hat{\rho}}{\partial t} \right)_{\rm rel} + \left(\frac{\partial \hat{\rho}}{\partial t} \right)_{\rm int} . \tag{20}$$

The interaction term $(\partial \rho/\partial t)_{\rm int}$ is intended to describe the deceleration of vehicles to the velocity of the next car ahead in situations when this moves slower and cannot be overtaken. Prigogine and co-workers [24,4] suggest describing processes of this kind by the *Boltzmann equation*

$$\left(\frac{\partial \rho}{\partial t} \right)_{\rm int} := \int_v^\infty dw\, (1-p) |v - w| \hat{\rho}(r, v, t) \hat{\rho}(r, w, t) \tag{21a}$$

$$- \int_0^v dw\, (1-p) |w - v| \hat{\rho}(r, w, t) \hat{\rho}(r, v, t) \tag{21b}$$

$$= (1-p) \hat{\rho}(r, v, t) \int_0^\infty dw\, (w - v) \hat{\rho}(r, w, t),$$

where p denotes the probability that a slower car can be overtaken. Functional relations for

$$p \equiv p(\rho, V, \Theta) \tag{22}$$

are proposed in Refs. [4,3,75]. The term (21a) corresponds to situations where a vehicle with speed $w > v$ must decelerate to speed v, causing an increase of phase-space density $\hat{\rho}(r, v, t)$. The rate of these situations is proportional (1) to the probability $(1 - p)$ that passing is not possible (which corresponds to the scattering cross section in kinetic gas theory), (2) to the relative velocity $|v - w|$ of the interacting vehicles, (3) to the phase-space density $\hat{\rho}(r, v, t)$ of vehicles which may hinder a vehicle with velocity $w > v$, and (4) to the phase-space density $\hat{\rho}(r, w, t)$ of vehicles with velocity $w > v$ that may be affected by an interaction. The term (21b) describes a decrease of phase-space density $\hat{\rho}(r, v, t)$ due to situations in which vehicles with velocity v must decelerate to a velocity $w < v$. A more detailed discussion of the interaction term (21) can be found in Refs. [4,2].

Note that the approach (21) assumes an instantaneous adaptation of velocity which does not take any braking time. Moreover, the deceleration process of the faster vehicle is assumed to happen at the location r of the slower vehicle, i.e., vehicles are implicitly modeled as pointlike objects without any space requirements. The first assumption is only justified for braking times that are short compared to temporal changes of phase-space density $\hat{\rho}$, but modifications for finite braking times are possible [75]. The second assumption is only acceptable for very small densities at which the average headway distance is much larger than the average vehicle length plus safe distance. It will, therefore, be corrected in Sec. VII. The corresponding modifications also implicitly take into account the pair correlations of succeeding vehicles [76]. These are neglected by the approach (21) due to its assumption of vehicular chaos, according to which the velocities of vehicles are not correlated until they interact with each other [2,1].

Now, we come to the description of acceleration processes by vehicles that do not move with their desired speeds. In this connection, Prigogine suggests a collective relaxation of the actual *velocity distribution*

$$P(v;r,t) := \frac{\hat{\rho}(r,v,t)}{\rho(r,t)} \tag{23}$$

towards an equilibrium velocity distribution $P_0(v)$ instead of an individual speed adjustment so that

$$\frac{dv}{dt} := 0\,. \tag{24}$$

In detail, Prigogine starts from the observation that free traffic is characterized by a certain velocity distribution $P_0(v)$ which corresponds to the distribution $P_0(v_0)$ of *desired velocities* v_0. Moreover, he assumes that the drivers' intention to get ahead with their desired speeds causes the phase-space density $\hat{\rho}(r,v,t)$ to approach the *equilibrium phase-space density*

$$\hat{\rho}_0(r,v,t) := \rho(r,t)P_0(v) \tag{25}$$

(exponentially) with a certain relaxation time τ which is given by the average duration of acceleration processes. Therefore, Prigogine's relaxation term has the form [24,25,4]

$$\left(\frac{\partial\hat{\rho}}{\partial t}\right)_{\rm rel} := \frac{\rho(r,t)P_0(v) - \hat{\rho}(r,v,t)}{\tau}\,. \tag{26}$$

Despite the merits of Prigogine's stimulating model, this approach has been severely criticized [2,51]. In a clear and detailed paper [2] Paveri-Fontana showed that Prigogine's model has a number of peculiar properties which are not compatible with empirical findings. For example, he demonstrates that the relaxation term (26) corresponds to *discontinuous* velocity changes which take place with a certain, time-dependent rate. Furthermore, Daganzo made the criticism that, according to (26), "the desired speed distribution is a property of the road and not the drivers" [51] which was already noted by Paveri-Fontana [2]. In reality, however, one can distinguish different "personalities" of drivers: "aggressive" ones desire to drive faster, "timid" ones slower. Therefore, Paveri-Fontana [2] developed an improved gas-kinetic traffic model which corrects the deficiencies of Prigogine's approach.

B. Paveri-Fontana's model

Paveri-Fontana assumes that each driver has an individual, characteristic desired velocity v_0. Consequently, the associated states $\vec{x}$ are given by place r, velocity v, and desired velocity v_0 so that Prigogine's phase-space density $\hat{\rho}(r,v,t)$ is replaced by $\hat{\rho}(r,v,v_0,t)$. The corresponding gas-kinetic equation (19) explicitly reads [77]

$$\frac{\partial\hat{\rho}}{\partial t} + \frac{\partial(\hat{\rho}v)}{\partial r} + \frac{\partial}{\partial v}\left(\hat{\rho}\frac{dv}{dt}\right) + \frac{\partial}{\partial v_0}\left(\hat{\rho}\frac{dv_0}{dt}\right) = \left(\frac{\partial\hat{\rho}}{\partial t}\right)_{\rm tr}\,. \tag{27a}$$

The term $\partial(\hat{\rho}dv_0/dt)/\partial v_0$ can be neglected since the desired velocity of each driver is normally time independent during a trip, which implies

$$\frac{dv_0}{dt} := 0\,. \tag{27b}$$

In contrast to Prigogine, Paveri-Fontana describes the acceleration towards the desired velocity v_0 by

$$\frac{dv}{dt} := \frac{1}{\tau}(v_0 - v) \tag{27c}$$

which means an *individual* instead of a *collective* relaxation. Relation (27c) can easily be replaced by other acceleration laws dv/dt or density-dependent driving programs as suggested by Alberti and Belli [26]. Alternatively, for acceleration processes an interaction approach can be formulated which was recently proposed by Nelson [1]. However, the assumption (27c) of exponential relaxation is a relatively good approximation since drivers gradually reduce the acceleration as they approach their desired velocity v_0.

Paveri-Fontana needs the transition term $(\partial\hat{\rho}/\partial t)_{\rm tr}$ only for the description of deceleration processes due to vehicular interactions. For these he assumes the Boltzmann equation [2]

$$\begin{aligned}\left(\frac{\partial\hat{\rho}}{\partial t}\right)_{\rm tr} := {} & (1-p)\int_v^\infty dw \int dw_0\, |v-w|\hat{\rho}(r,v,w_0,t)\hat{\rho}(r,w,v_0,t) \\ & -(1-p)\int_0^v dw \int dw_0\, |w-v|\hat{\rho}(r,w,w_0,t)\hat{\rho}(r,v,v_0,t)\end{aligned} \tag{27d}$$

which has an analogous interpretation to (21). (For details cf. Ref. [2].) Note that, according to (27d), "the velocity of the slow car is unaffected by the interaction or by the fact of being passed" [2] and that "no driver changes his desired speed" [2] during interactions. Therefore, the interaction term (27d) fulfills the requirements called for by Daganzo [51]

(1) that "a car is an anisotropic particle that mostly responds to frontal stimuli" [51] and that "a slow car should be virtually unaffected by its interaction with faster cars passing it (or queueing behind it)" [51];

(2) that "interactions do not change the 'personality' (aggressive/timid) of any car" [51].

Finally, note that the proportion of vehicles jamming behind slower cars cannot accelerate. This circumstance can be taken into account by a density and maybe velocity or variance dependence of the relaxation time [4,3,75]:

$$\tau \equiv \tau(\rho, V, \Theta) . \tag{28}$$

In order to compare Paveri-Fontana's traffic equation with Prigogine's we integrate Eq. (27) with respect to v_0 and obtain the *reduced Paveri-Fontana equation*

$$\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial (v\tilde{\rho})}{\partial r} + \frac{\partial}{\partial v}\left[\tilde{\rho}(r,v,t)\frac{\tilde{V}_0(v;r,t) - v}{\tau}\right] = (1-p)\tilde{\rho}(r,v,t)\int_0^\infty dw\,(w-v)\tilde{\rho}(r,w,t) . \tag{29}$$

Here, we have introduced the *reduced phase-space density*

$$\tilde{\rho}(r,v,t) := \int dv_0\,\hat{\rho}(r,v,v_0,t) \tag{30}$$

and the quantity

$$\tilde{V}_0(v;r,t) := \int dv_0\,v_0\frac{\hat{\rho}(r,v,v_0,t)}{\tilde{\rho}(r,v,t)} . \tag{31}$$

The only difference with respect to Prigogine's formulation (20)–(26) is obviously the other relaxation term.

IV. DERIVATION OF MACROSCOPIC TRAFFIC EQUATIONS

Since we are mainly interested in the temporal evolution of collective (macroscopic) quantities like the spatial density

$$\rho(r,t) := \int dv\,\tilde{\rho}(r,v,t) \tag{32}$$

per lane, the average velocity

$$V(r,t) \equiv \langle v\rangle := \int dv\,v\frac{\tilde{\rho}(r,v,t)}{\rho(r,t)} , \tag{33}$$

and the velocity variance

$$\Theta(r,t) \equiv \langle[v - V(r,t)]^2\rangle := \int dv\,[v-V(r,t)]^2\frac{\tilde{\rho}(r,v,t)}{\rho(r,t)} = \langle v^2\rangle - [V(r,t)]^2 \tag{34}$$

we will now derive equations for the moments $m_{k,0}$ with

$$m_{k,l}(r,t) \equiv \rho(r,t)\langle v^k (v_0)^l\rangle := \int dv \int dv_0\,v^k (v_0)^l \hat{\rho}(r,v,v_0,t) . \tag{35}$$

By multiplying Paveri-Fontana's equation (29) with v^k and integrating with respect to v we obtain [2], via partial integration,

$$\frac{\partial}{\partial t}m_{k,0} + \frac{\partial}{\partial r}m_{k+1,0} + \int dv\,v^k\frac{\partial}{\partial v}\left(\tilde{\rho}\frac{\tilde{V}_0(v)-v}{\tau}\right)$$

$$= \frac{\partial}{\partial t}m_{k,0} + \frac{\partial}{\partial r}m_{k+1,0} - \int dv\,kv^{k-1}\left(\tilde{\rho}\frac{\tilde{V}_0(v)-v}{\tau}\right)$$

$$= \frac{\partial}{\partial t}m_{k,0} + \frac{\partial}{\partial r}m_{k+1,0} - \frac{k}{\tau}(m_{k-1,1} - m_{k,0}) \tag{36a}$$

$$= (1-p)\int dv\,\tilde{\rho}(r,v,t)\int dw\,(wv^k - v^{k+1})\tilde{\rho}(r,w,t)$$

$$= (1-p)(m_{1,0}m_{k,0} - m_{k+1,0}m_{0,0}) . \tag{36b}$$

Applying the analogous procedure to Prigogine's model (20)–(26), for the moments

$$m_{k,0}(r,t) \equiv \rho(r,t)\langle v^k\rangle := \int dv\,v^k\hat{\rho}(r,v,t) \tag{37}$$

one can derive the equations

$$\frac{\partial}{\partial t}m_{k,0} + \frac{\partial}{\partial r}m_{k+1,0} = \frac{1}{\tau}(m_{0,k} - m_{k,0}) + (1-p)(m_{1,0}m_{k,0} - m_{k+1,0}m_{0,0}) \tag{38}$$

(cf. [2]) where

$$m_{0,k}(r,t) := \int dv_0\,(v_0)^k\hat{\rho}_0(r,v_0,t) = \rho(r,t)\int dv_0\,(v_0)^k P_0(v_0) . \tag{39}$$

A comparison of moment equations (36) with (38) shows that Prigogine's and Paveri-Fontana's model lead to identical equations for spatial density $\rho(r,t) = m_{0,0}(r,t)$ and average velocity $V(r,t) = m_{1,0}(r,t)/\rho(r,t)$, despite the different approaches for the relaxation term. However, the equations for higher-order moments $m_{k,0}(r,t)$ with $k \geq 2$ differ.

Obviously, Eqs. (36) as well as (38) represent a hierarchy of nonclosed equations since the equation for the kth moment $m_{k,0}$ depends on the $(k+1)$st moment $m_{k+1,0}$. As a consequence, the density equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho V)}{\partial r} = 0 \tag{40}$$

depends on the average velocity V, the velocity equation

$$\begin{aligned}\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial r} &= -\frac{1}{\rho}\frac{\partial(\rho\Theta)}{\partial r} + \frac{1}{\tau}(V_0 - V) - (1-p)\rho\Theta \\ &= -\frac{1}{\rho}\frac{\partial \mathcal{P}}{\partial r} + \frac{1}{\tau}[V_e(\rho,V,\Theta) - V]\end{aligned} \tag{41}$$

on variance Θ, etc. Here, we have introduced the *average desired velocity*

$$V_0(r,t) := \int dv \int dv_0\, v_0 \frac{\hat{\rho}(r,v,v_0,t)}{\rho(r,t)}\,, \tag{42}$$

the so-called *traffic pressure* [25,3,71]

$$\begin{aligned}\mathcal{P}(r,t) &:= \frac{1}{\rho(r,t)}\int dv\,(v-V)\tilde{\rho}(r,v,t) \\ &\quad\times \int dw\,(v-w)\tilde{\rho}(r,w,t) \\ &= \int dv\,(v-V)^2\tilde{\rho}(r,v,t) = \rho(r,t)\Theta(r,t)\,,\end{aligned} \tag{43}$$

and the *equilibrium velocity*

$$V_e(\rho,V,\Theta) := V_0 - \tau(\rho,V,\Theta)[1-p(\rho,V,\Theta)]\mathcal{P} \tag{44}$$

which is related to stationary and spatially homogeneous traffic flow.

Equations (40) and (41) are easily derivable from the moment equations (36) and (38), respectively, by use of $m_{0,0} = \rho$ and

$$\frac{\partial m_{1,0}}{\partial t} = \frac{\partial(\rho V)}{\partial t} = \rho\frac{\partial V}{\partial t} + V\frac{\partial \rho}{\partial t}\,. \tag{45}$$

The *variance equation* is obtained analogously. For the traffic equation of Paveri-Fontana it reads

$$\begin{aligned}\frac{\partial \Theta}{\partial t} + V\frac{\partial \Theta}{\partial r} &= -2\Theta\frac{\partial V}{\partial r} - \frac{1}{\rho}\frac{\partial(\rho\Gamma)}{\partial r} \\ &\quad + \frac{2}{\tau}(\mathcal{C} - \Theta) - (1-p)\rho\Gamma \\ &= -\frac{2\mathcal{P}}{\rho}\frac{\partial V}{\partial r} - \frac{1}{\rho}\frac{\partial \mathcal{J}}{\partial r} \\ &\quad + \frac{2}{\tau}[\Theta_e(\rho,V,\Theta,\mathcal{C},\mathcal{J}) - \Theta]\end{aligned} \tag{46}$$

and depends on the covariance

$$\begin{aligned}\mathcal{C}(r,t) &\equiv \langle (v-V)(v_0-V_0)\rangle \\ &:= \int dv_0 \int dv\,(v-V)(v_0-V_0)\frac{\hat{\rho}(r,v,v_0,t)}{\rho(r,t)} \\ &= \int dv\,(v-V)[\tilde{V}_0(v) - V_0]\frac{\tilde{\rho}(r,v,t)}{\rho(r,t)}\end{aligned} \tag{47}$$

as well as the third central moment

$$\Gamma(r,t) \equiv \langle (v-V)^3\rangle := \int dv\,(v-V)^3\frac{\tilde{\rho}(r,v,t)}{\rho(r,t)}\,. \tag{48}$$

In addition, we have introduced the *flux density of velocity variance*

$$\begin{aligned}\mathcal{J}(r,t) &:= \frac{1}{\rho(r,t)}\int dv\,(v-V)^2\tilde{\rho}(r,v,t) \\ &\quad\times \int dw\,(v-w)\tilde{\rho}(r,w,t) \\ &= \int dv\,(v-V)^3\tilde{\rho}(r,v,t) = \rho(r,t)\Gamma(r,t)\end{aligned} \tag{49}$$

(which corresponds to the "heat flow" in conventional fluid dynamics) and the *equilibrium variance*

$$\Theta_e(\rho,V,\Theta,\mathcal{C},\mathcal{J}) := \mathcal{C} - \frac{\tau(\rho,V,\Theta)}{2}[1-p(\rho,V,\Theta)]\mathcal{J}\,. \tag{50}$$

A. Approximate closed macroscopic traffic equations

We will now face the problem of closing the hierarchy of moment equations by a suitable approximation. The simplest approximations replace a macroscopic quantity $Q(r,t)$ (which would be determined by a dynamic equation) by its equilibrium value Q_e which belongs to the stationary and spatially homogeneous solution. Approximations of this kind are zeroth-order approximations. The simplest one is obtained by a substitution of $V(r,t)$ [which actually obeys Eq. (41)] by the equilibrium velocity

$$V_e(\rho) := V_0 - \tau(\rho)[1-p(\rho)]\rho\Theta_e(\rho) \tag{51}$$

[cf. (44)]. Equations (40) and (51) obviously correspond to the model (1) and (2) of Lighthill and Whitham. Relation (51) specifies the equilibrium velocity-density relation (2) in accordance with Paveri-Fontana's traffic equation. It could be interpreted as a theoretical result concerning the dependence of $V_e(\rho)$ on the microscopic processes of traffic flow: According to (51), the equilibrium velocity $V_e(\rho)$ is given by the average desired velocity V_0 diminished by a term arising from necessary deceleration maneuvers due to interactions of vehicles.

However, according to Eq. (41), the approximation $V(r,t) \approx V_e[\rho(r,t)]$ is only justified for $\tau \to 0$ which is not compatible with empirical data. Consequently, the latter does not adequately describe nonequilibrium situations like on-ramp traffic or stop-and-go traffic where the velocity is not uniquely given by the spatial density $\rho(r,t)$.

Another zeroth-order approximation is found by leaving Eq. (41) unchanged but replacing the dynamic variance $\Theta(r,t)$ by the equilibrium variance

$$\Theta_e(\rho,V) := \mathcal{C}_e(\rho,V) - \frac{\tau(\rho,V)}{2}[1-p(\rho,V)]\rho\Gamma_e(\rho,V) \tag{52}$$

[cf. (50)]. (Here, the subscript e again indicates the equilibrium value or relation of a function.) The resulting model (40), (41), and (52) obviously corresponds to the model (1) and (11) of Phillips, this time specifying the equilibrium variance-density relation in accordance with Paveri-Fontana's traffic model. A complete agreement between (52) and (11b) results for $\mathcal{C}_e(\rho, V) \equiv \mathcal{C}_e(\rho)$, $\Gamma_e(\rho, V) \equiv \Gamma_e(\rho)$, and a special choice of the functional relation $\tau(\rho, V)[1-p(\rho, V)] \equiv \tau(\rho)[1-p(\rho)]$.

However, it is not fully justified to assume that the variance $\Theta(r,t)$ is always in equilibrium $\Theta_e(\rho, V)$, since the corresponding relaxation time $2/\tau$ is of the order of the relaxation time $1/\tau$ for the velocity $V(r,t)$. Moreover, the approximation $\Theta(r,t) \approx \Theta_e[\rho(r,t), V(r,t)]$ does not describe the empirically observed increase of variance Θ directly before a traffic jam develops [43,5]. Therefore we also need the dynamic variance equation (46). The remaining problem is how to obtain suitable relations for $\Gamma(r,t)$ and $\mathcal{C}(r,t)$.

B. Euler-like traffic equations

Before looking for dynamic relations for $\Gamma(r,t)$ and $\mathcal{C}(r,t)$, it is plausible first to look for equilibrium relations which apply to stationary and spatially homogeneous traffic. For this purpose we require the equilibrium solution $\hat{\rho}_e(v, v_0)$ of Paveri-Fontana's traffic equation (27).

Unfortunately, it seems impossible to find an analytical expression for $\hat{\rho}_e(v, v_0)$, but in order to derive equations for the velocity moments $\langle v^k \rangle$ we are mainly interested in, it is sufficient to find the stationary and spatially homogeneous solution $\tilde{\rho}_e(v)$ of the *reduced* Paveri-Fontana equation (29). For this we need to know the relation

$$\tilde{V}_0(v) = a_0 + a_1\,\delta v + a_2\,(\delta v)^2 + \cdots + a_n\,(\delta v)^n \tag{53}$$

with

$$\delta v := v - V \tag{54}$$

and arbitrary n. However, the equation that determines $\tilde{V}_0(v)$ depends on the unknown quantity

$$\tilde{\Theta}_0(v) := \int dv_0\,(v_0 - V_0)^2 \frac{\hat{\rho}_e(v, v_0)}{\tilde{\rho}_e(v)}, \tag{55}$$

etc., so that we are again confronted with a nonclosed hierarchy of equations.

Luckily, from empirical data and microsimulations we know that the equilibrium velocity distribution

$$P_e(v) := \frac{\tilde{\rho}_e(v)}{\rho_e} \tag{56}$$

(at least in the range of stable traffic without stop-and-go waves) is approximately a Gaussian distribution [60,61,3,62,33]:

$$P_e(v) = \frac{1}{\sqrt{2\pi\Theta_e}}\, e^{-(v-V_e)^2/(2\Theta_e)}\,. \tag{57}$$

Inserting (53) and (57) into the equation

$$\frac{\partial}{\partial v}\left(\tilde{\rho}_e \frac{\tilde{V}_0(v) - v}{\tau}\right) = -(1-p)\tilde{\rho}_e \rho_e\,\delta v \tag{58}$$

which corresponds to Eq. (29) in the stationary and spatially homogeneous case, we find the condition

$$\begin{aligned}
\frac{\partial}{\partial v}\left(\tilde{\rho}_e \frac{\tilde{V}_0(v) - v}{\tau}\right) &= \frac{\tilde{V}_0(v) - v}{\tau}\frac{\partial \tilde{\rho}_e}{\partial v} + \frac{\tilde{\rho}_e}{\tau}\left(\frac{\partial \tilde{V}_0(v)}{\partial v} - 1\right) \\
&= \frac{\tilde{\rho}_e}{\tau}\Bigg[(a_1 - 1) + \left(2a_2 - \frac{a_0 - V_e}{\Theta_e}\right)\delta v + \left(3a_3 - \frac{a_1 - 1}{\Theta_e}\right)(\delta v)^2 \cdots \\
&\quad + \left(k a_k - \frac{a_{k-2}}{\Theta_e}\right)(\delta v)^{k-1} \cdots - \frac{a_{n-1}}{\Theta_e}(\delta v)^n - \frac{a_n}{\Theta_e}(\delta v)^{n+1}\Bigg] \qquad \text{(59a)} \\
&\overset{!}{=} -(1-p)\tilde{\rho}_e\rho_e\,\delta v\,. \qquad \text{(59b)}
\end{aligned}$$

A comparison of the coefficients of $(\delta v)^k$ in (59a) and (59b) leads to

$$a_n = 0\,, \quad a_{n-1} = 0\,, \quad \cdots \quad a_2 = 0\,, \quad a_1 = 1\,, \tag{60}$$

and

$$a_0 = V_e + \tau(1-p)\rho_e\Theta_e = V_0\,, \tag{61}$$

where we have utilized relation (44) with (43). Consequently, for equilibrium situations the velocity distribution (57) implies

$$\tilde{V}_0(v) = V_0 + \delta v\,. \tag{62}$$

With (57) and (62) we can now derive equilibrium relations for $\mathcal{C}$ and Γ. One obtains

$$\Gamma_e = 0 \tag{63}$$

and

$$\mathcal{C}_e = \Theta_e\,. \tag{64}$$

Next, we are looking for relations for nonequilibrium cases. Assuming that the velocity distribution

$$P(v;r,t) := \frac{\tilde{\rho}(r,v,t)}{\rho(r,t)} \quad (65)$$

locally approaches the equilibrium distribution $P_e[V(r,t),\Theta(r,t)]$ very rapidly, we can apply the zeroth-order *approximation of local equilibrium:*

$$\begin{aligned} P(v;r,t) &\approx P_{(0)}(v;r,t) \\ &:= P_e[V(r,t),\Theta(r,t)] \\ &= \frac{1}{\sqrt{2\pi\Theta(r,t)}} e^{-[v-V(r,t)]^2/[2\Theta(r,t)]} . \end{aligned} \quad (66)$$

Furthermore, in order to fulfill the compatibility condition

$$\mathcal{C}(r,t) = \int dv [v - V(r,t)][v_0 - \tilde{V}_0(v;r,t)] P(v;r,t) \quad (67)$$

[cf. (47)], we must generalize relation (62) to

$$\tilde{V}_0(v;r,t) = V_0 + \frac{\mathcal{C}(r,t)}{\Theta(r,t)} \delta v \quad (68)$$

which is fully consistent with (64). Relations (66) and (68) yield zeroth-order relations for the spatiotemporal variation of $\mathcal{C}(r,t)$ and $\mathcal{J}(r,t)$. For the flux density of the velocity variance we find

$$\mathcal{J}(r,t) \approx \mathcal{J}_{(0)}(\rho,V,\Theta) = \rho\Gamma_{(0)}(\rho,V,\Theta) = 0 , \quad (69)$$

whereas for the covariance the dynamic equation

$$\frac{\partial \mathcal{C}}{\partial t} + V\frac{\partial \mathcal{C}}{\partial r} = -\mathcal{C}\frac{\partial V}{\partial r} - \Theta\frac{\partial V_0}{\partial r} + \frac{1}{\tau}(\Theta_0 - \mathcal{C}) - 2(1-p)\rho\mathcal{C}\sqrt{\frac{\Theta}{\pi}} \quad (70)$$

can be derived from the Paveri-Fontana equation (27) since

$$\begin{aligned} &\int dv \int dv_0\, (\delta v)^2 \delta v_0 \hat{\rho}(r,v,v_0,t) \\ &\quad = \int dv\, (\delta v)^2 [\tilde{V}_0(v) - V_0] \tilde{\rho}(r,v,t) \\ &\quad = \int dv\, (\delta v)^3 \frac{\mathcal{C}}{\Theta} \tilde{\rho}(r,v,t) = \mathcal{J}\frac{\mathcal{C}}{\Theta} \end{aligned} \quad (71)$$

($\delta v_0 := v_0 - V_0$). (The somewhat lengthy but straightforward calculation is presented in Ref. [79].)

In the zeroth-order covariance equation (70) the quantity

$$\Theta_0(r,t) := \int dv \int dv_0\, [v_0 - V_0(r,t)]^2 \frac{\hat{\rho}(r,v,v_0,t)}{\rho(r,t)} \quad (72)$$

denotes the *variance of desired velocities.* The term $-\Theta\partial V_0/\partial r$ normally vanishes since the average desired velocity V_0 is approximately constant almost everywhere (cf. [77]). Due to (64), the equilibrium variance related to stationary and homogeneous traffic is obviously determined by the implicit relation

$$\begin{aligned} \Theta_e(\rho_e, V_e, \Theta_e) &= \mathcal{C}_e(\rho_e, V_e, \Theta_e) \\ &= \Theta_0 - 2\tau(1-p)\rho_e\Theta_e\sqrt{\frac{\Theta_e}{\pi}} . \end{aligned} \quad (73)$$

Inserting the above results into Eqs. (40), (41), and (46), we obtain the following zeroth-order approximations of the density, velocity, and variance equations respectively:

$$\frac{\partial \rho}{\partial t} + V\frac{\partial \rho}{\partial r} = -\rho\frac{\partial V}{\partial r} , \quad (74)$$

$$\begin{aligned} \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial r} &= -\frac{1}{\rho}\frac{\partial(\rho\Theta)}{\partial r} + \frac{1}{\tau}(V_0 - V) - (1-p)\rho\Theta \\ &= -\frac{1}{\rho}\frac{\partial \mathcal{P}}{\partial r} + \frac{1}{\tau}[V_e(\rho,V,\Theta) - V] , \end{aligned} \quad (75)$$

$$\begin{aligned} \frac{\partial \Theta}{\partial t} + V\frac{\partial \Theta}{\partial r} &= -2\Theta\frac{\partial V}{\partial r} + \frac{2}{\tau}(\mathcal{C} - \Theta) \\ &= -\frac{2\mathcal{P}}{\rho}\frac{\partial V}{\partial r} + \frac{2}{\tau}(\mathcal{C} - \Theta) . \end{aligned} \quad (76)$$

Equations (74), (75), and (76) are the *Euler-like equations* of vehicular traffic [58]. In comparison with the Euler equations for ordinary fluids [52–55] they contain additional terms.

(1) The terms $(V_0 - V)/\tau$ and $2(\mathcal{C} - \Theta)/\tau$ arise from the acceleration of vehicles towards the drivers' desired velocities v_0, i.e., they are a consequence of the fact that driver-vehicle units are *active* systems.

(2) The term $-(1-p)\rho\Theta$ results from the vehicles' interactions. It would vanish if momentum were a collisional invariant during vehicular interactions as is the case for atomic collisions [74]. However, without this term the "vehicular fluid" would speed up at bottlenecks which is, of course, unrealistic.

Moreover, the covariance equation (70) is a complementary equation which arises from the drivers' tendency to move with their desired velocities v_0.

C. Equilibrium relations and fundamental diagram

For vehicular traffic, the only dynamic quantity that remains unchanged in a closed system (i.e., a circular road) is the average spatial density $\bar{\rho}$ (due to the conservation of the number of vehicles). As a consequence, the equilibrium traffic situation is uniquely determined by $\bar{\rho}$ which obviously agrees with the equilibrium density ρ_e. Equilibrium relations for the average velocity $V_e(\rho_e)$ and the velocity variance $\Theta_e(\rho_e)$ in dependence on $\rho_e = \bar{\rho}$ can be obtained from Eqs. (44) and (73) if the relations $p(\rho,V,\Theta)$ and $\tau(\rho,V,\Theta)$ are given (cf. [4,3]). A simple procedure for finding a solution of these implicit equations is to numerically integrate the equations

$$\begin{aligned} \frac{dV}{dy} &= V_e(\rho_e, V(y), \Theta(y)) - V(y) \\ &= V_0 - \tau(\rho_e,V,\Theta)[1 - p(\rho_e,V,\Theta)]\varrho_e\Theta - V , \end{aligned} \quad (77)$$

$$\frac{d\Theta}{dy} = \Theta_e(\rho_e, V(y), \Theta(y)) - \Theta(y)$$
$$= \Theta_0 - 2\tau(\rho_e, V, \Theta)[1 - p(\rho_e, V, \Theta)]\varrho_e \Theta \sqrt{\frac{\Theta}{\pi}} - \Theta \tag{78}$$

until $dV/dy = 0$ and $d\Theta/dy = 0$. Here, we have replaced ρ_e by $\varrho_e = \varrho_e(\rho_e, V)$ in accordance with Sec. VII B in order to take into account the space requirements of vehicles. The theoretical results for the equilibrium velocity-density relation $V_e(\rho_e) = \lim_{y\to\infty} V(y)$, the equilibrium variance-density relation $\Theta_e(\rho_e) = \lim_{y\to\infty} \Theta(y)$, and the *fundamental diagram*

$$q_e(\rho_e) := \rho_e V_e(\rho_e) \tag{79}$$

can be directly compared with empirical data.

If, however, $p(\rho, V, \Theta)$ or $\tau(\rho, V, \Theta)$ is an unknown relation, it is still possible to derive from the fundamental diagram $q_e(\rho_e)$ the equilibrium variance-density relation $\Theta_e(\rho_e)$ for which an empirical relation seems to be missing. From (77) and (79) we get

$$\tau(1-p)\varrho_e\Theta_e(\rho_e) = V_0 - V_e(\rho_e) = V_0 - \frac{q_e(\rho_e)}{\rho_e}. \tag{80}$$

Inserting this into (73) we find

$$\Theta_e(\rho_e) = \Theta_0 - 2\tau(1-p)\varrho_e\Theta_e(\rho_e)\sqrt{\frac{\Theta_e(\rho_e)}{\pi}}$$
$$= \Theta_0 - 2[V_0 - V_e(\rho_e)]\sqrt{\frac{\Theta_e(\rho_e)}{\pi}}. \tag{81}$$

This results in a quadratic equation for the standard deviation $\sqrt{\Theta_e(\rho_e)}$ of vehicle velocities which is solved by

$$\sqrt{\Theta_e(\rho_e)} = -\frac{V_0 - V_e(\rho_e)}{\sqrt{\pi}} + \sqrt{\frac{[V_0 - V_e(\rho_e)]^2}{\pi} + \Theta_0}. \tag{82}$$

V. APPROXIMATE SOLUTION OF PAVERI-FONTANA'S TRAFFIC EQUATION

The traffic equation of Paveri-Fontana was mathematically investigated in several papers dealing with the existence, uniqueness, and numerical determination of a solution which satisfies the nonlinear initial-value boundary problem [80–82]. However, the approximate dynamic solution of the reduced Paveri-Fontana equation (29) which will be presented in this section has not been proposed before to our knowledge.

As one would expect, in nonequilibrium situations the zeroth-order approximation (66) does not solve the reduced Paveri-Fontana equation (29) exactly. Therefore, we write

$$\tilde{\rho}(r, v, t) =: \tilde{\rho}_{(0)}(r, v, t) + \tilde{\rho}_{(1)}(r, v, t) \tag{83}$$

with

$$\tilde{\rho}_{(0)}(r, v, t) := \rho(r, t) P_{(0)}(v; r, t)$$
$$= \frac{\rho(r,t)}{\sqrt{2\pi\Theta(r,t)}} e^{-[v - V(r,t)]^2/[2\Theta(r,t)]} \tag{84}$$

and try to derive a relation for the deviation $\tilde{\rho}_{(1)}(r, v, t)$. Utilizing the fact that the correction term $\tilde{\rho}_{(1)}(r, v, t)$ will usually be small compared to $\tilde{\rho}_{(0)}(r, v, t)$ we have

$$\tilde{\rho}_{(1)}(r, v, t) \ll \tilde{\rho}_{(0)}(r, v, t) \tag{85}$$

and we get

$$\frac{\partial\tilde{\rho}}{\partial t} + v\frac{\partial\tilde{\rho}}{\partial r} + \frac{\partial}{\partial v}\left(\tilde{\rho}\frac{\tilde{V}_0(v) - v}{\tau}\right) \approx \frac{\partial\tilde{\rho}_{(0)}}{\partial t} + v\frac{\partial\tilde{\rho}_{(0)}}{\partial r} + \frac{\partial}{\partial v}\left(\tilde{\rho}_{(0)}\frac{\tilde{V}_0(v) - v}{\tau}\right)$$
$$= \frac{\partial\tilde{\rho}_{(0)}}{\partial t} + v\frac{\partial\tilde{\rho}_{(0)}}{\partial r} + \frac{\tilde{V}_0(v) - v}{\tau}\frac{\partial\tilde{\rho}_{(0)}}{\partial v} + \frac{\tilde{\rho}_{(0)}}{\tau}\left(\frac{\partial\tilde{V}_0(v)}{\partial v} - 1\right). \tag{86}$$

(For a detailled discussion of this approximation cf. [52,53,55].) Now, introducing the abbreviation

$$\frac{d}{dt} := \frac{\partial}{\partial t} + v\frac{\partial}{\partial r} \tag{87}$$

we can write

$$\frac{\partial\tilde{\rho}_{(0)}}{\partial t} + v\frac{\partial\tilde{\rho}_{(0)}}{\partial r} = \frac{d\tilde{\rho}_{(0)}}{dt}$$
$$= \frac{\partial\tilde{\rho}_{(0)}}{\partial\rho}\frac{d\rho}{dt} + \frac{\partial\tilde{\rho}_{(0)}}{\partial V}\frac{dV}{dt} + \frac{\partial\tilde{\rho}_{(0)}}{\partial\Theta}\frac{d\Theta}{dt}$$
$$= \frac{\tilde{\rho}_{(0)}}{\rho}\frac{d\rho}{dt} + \frac{\tilde{\rho}_{(0)}}{\Theta}\delta v\frac{dV}{dt}$$
$$+ \frac{\tilde{\rho}_{(0)}}{2\Theta}\left(\frac{(\delta v)^2}{\Theta} - 1\right)\frac{d\Theta}{dt}. \tag{88}$$

Relations for $d\rho/dt$, dV/dt, and $d\Theta/dt$ can be obtained from the Euler-like equations (74), (75), and (76) via

$$\frac{d}{dt} = \frac{\partial}{\partial t} + V\frac{\partial}{\partial r} + \delta v\frac{\partial}{\partial r}. \tag{89}$$

We find

$$\frac{d\rho}{dt} = \delta v\frac{\partial\rho}{\partial r} - \rho\frac{\partial V}{\partial r}, \tag{90a}$$

$$\frac{dV}{dt} = \delta v\frac{\partial V}{\partial r} - \frac{1}{\rho}\frac{\partial(\rho\Theta)}{\partial r} + \frac{1}{\tau}[V_e(\rho, V, \Theta) - V], \tag{90b}$$

and

$$\frac{d\Theta}{dt} = \delta v\frac{\partial\Theta}{\partial r} - 2\Theta\frac{\partial V}{\partial r} + \frac{2}{\tau}(\mathcal{C} - \Theta). \tag{90c}$$

For the interaction term we apply a linear approximation in $\tilde{\rho}_{(1)}(r,v,t)$ which is justified by relation (85). The result is

$$(1-p)\tilde{\rho}(r,v,t)\int dw\,(w-v)\tilde{\rho}(r,w,t)$$

$$\approx (1-p)\tilde{\rho}_{(0)}(r,v,t)\rho(V-v) - \int dw\, L(v,w;r,t)\tilde{\rho}_{(1)}(r,w,t) \qquad (91a)$$

where we have introduced a linear operator $\underline{L}$ with the components

$$L(v,w;r,t) := (1-p)\rho(r,t)\{[v-V(r,t)]\delta(v-w) + P_{(0)}(v;r,t)(v-w)\}\,. \qquad (91b)$$

Here, $\delta(v-w)$ denotes Dirac's delta function. The linear operator $\underline{L}$ possesses an infinite number of eigenvalues $1/\tau_\mu$ (cf. [55,83–86]). The relevant eigenvalue is the smallest one since it characterizes temporal changes that take place on the time scale we are interested in. It is of the order of the average *interaction rate* per vehicle [53,52,54]

$$\begin{aligned}\frac{1}{\tau_0} &:= \frac{1-p}{\rho(r,t)}\int dv \int_{w<v} dw\,|w-v|\tilde{\rho}(r,w,t)\tilde{\rho}(r,v,t) \\ &\approx (1-p)\rho(r,t)\int dv \\ &\quad\times \int_{w<v} dw\,|w-v|P_{(0)}(w;r,t)P_{(0)}(v;r,t) \\ &= (1-p)\rho(r,t)\sqrt{\frac{\Theta}{\pi}}\,. \end{aligned} \qquad (91c)$$

The other eigenvalues are somewhat larger [55,83–86] (i.e., $\tau_\mu < \tau_0$ for $\mu \neq 0$) and they describe fast fluctuations which can be *adiabatically eliminated* [78]. As a consequence, we can make the so-called *relaxation time approximation* [87]

$$\int dw\, L(v,w;r,t)\tilde{\rho}_{(1)}(r,w,t) \approx \frac{\tilde{\rho}_{(1)}(r,v,t)}{\tau_0}\,. \qquad (91d)$$

Now, we calculate

$$\begin{aligned}&\frac{\tilde{V}_0(v)-v}{\tau}\frac{\partial\tilde{\rho}_{(0)}}{\partial v} + \frac{\tilde{\rho}_{(0)}}{\tau}\left(\frac{\partial\tilde{V}_0(v)}{\partial v}-1\right) - (1-p)\tilde{\rho}_{(0)}\rho(V-v) \\ &= \frac{1}{\tau}\left(V_0 + \frac{\mathcal{C}}{\Theta}\delta v - v\right)\left(-\frac{\tilde{\rho}_{(0)}}{\Theta}\delta v\right) + \frac{\tilde{\rho}_{(0)}}{\tau}\left(\frac{\mathcal{C}}{\Theta}-1\right) + (1-p)\tilde{\rho}_{(0)}\rho\,\delta v \\ &= \frac{\tilde{\rho}_{(0)}}{\tau\Theta}\left[(\mathcal{C}-\Theta)\left(1-\frac{(\delta v)^2}{\Theta}\right) - (V_e - V)\delta v\right]\,. \end{aligned} \qquad (92)$$

Inserting (86), (88), and (90)–(92) into the reduced Paveri-Fontana equation (29) we finally obtain

$$\begin{aligned}\tilde{\rho}_{(1)}(r,v,t) \approx -\tau_0 \Bigg\{ &\frac{\tilde{\rho}_{(0)}}{\rho}\left(\delta v\frac{\partial\rho}{\partial r} - \rho\frac{\partial V}{\partial r}\right) + \frac{\tilde{\rho}_{(0)}}{\Theta}\delta v\left(\delta v\frac{\partial V}{\partial r} - \frac{\Theta}{\rho}\frac{\partial\rho}{\partial r} - \frac{\partial\Theta}{\partial r} + \frac{1}{\tau}(V_e - V)\right) \\ &+ \frac{\tilde{\rho}_{(0)}}{2\Theta}\left(\frac{(\delta v)^2}{\Theta}-1\right)\left(\delta v\frac{\partial\Theta}{\partial r} - 2\Theta\frac{\partial V}{\partial r} + \frac{2}{\tau}(\mathcal{C}-\Theta)\right) - \frac{\tilde{\rho}_{(0)}}{\Theta}\left[\frac{\mathcal{C}-\Theta}{\tau}\left(\frac{(\delta v)^2}{\Theta}-1\right) + \frac{V_e - V}{\tau}\delta v\right]\Bigg\} \\ = -\tilde{\rho}_{(0)}\tau_0 &\left(\frac{(\delta v)^3}{2\Theta^2} - \frac{3\,\delta v}{2\Theta}\right)\frac{\partial\Theta}{\partial r}\,. \end{aligned} \qquad (93)$$

Obviously, the correction term $\tilde{\rho}_{(1)}(r,v,t)$ is a consequence of the finite *interaction free time* τ_0 which causes a delayed adjustment of $\tilde{\rho}(r,v,t)$ to the local equilibrium $\tilde{\rho}_{(0)}(r,v,t)$. However, in order to take into account the effects of finite reaction time and braking time we must add a time period $\tau' > 0$ to the interaction free time τ_0. Hence, τ_0 must be replaced by the *adaptation time*

$$\tau_* = \tau_0 + \tau'\,. \qquad (94)$$

VI. NAVIER-STOKES-LIKE TRAFFIC EQUATIONS

With the corrected phase-space density

$$\begin{aligned}\tilde{\rho}(r,v,t) &\approx \tilde{\rho}_{(0)}(r,v,t) + \tilde{\rho}_{(1)}(r,v,t) \\ &\approx \tilde{\rho}_{(0)}(r,v,t)\left[1 - \tau_*\left(\frac{(\delta v)^3}{2\Theta^2} - \frac{3\,\delta v}{2\Theta}\right)\frac{\partial\Theta}{\partial r}\right] \end{aligned} \qquad (95)$$

we can calculate corrected relations for the collective (macroscopic) quantities

$$F(r,t) \equiv \langle f(v) \rangle := \int dv\, f(v) \frac{\tilde{\rho}(r,v,t)}{\rho(r,t)} \approx F_{(0)}(r,t) + F_{(1)}(r,t) \tag{96}$$

where

$$F_{(i)}(r,t) \equiv \langle f(v) \rangle_{(i)} := \int dv\, f(v) \frac{\tilde{\rho}_{(i)}(r,v,t)}{\rho(r,t)} . \tag{97}$$

We find

$$\rho(r,t) \approx \rho_{(0)}(r,t) , \quad V(r,t) \approx V_{(0)}(r,t) ,$$
$$\Theta(r,t) \approx \Theta_{(0)}(r,t) , \quad \mathcal{P}(r,t) \approx \mathcal{P}_{(0)}(r,t) , \tag{98}$$

and

$$\mathcal{C}(r,t) \approx \mathcal{C}_{(0)}(r,t) . \tag{99}$$

However, for the flux density of the velocity variance we get

$$\mathcal{J}(r,t) \approx \mathcal{J}_{(1)}(\rho, V, \Theta) \equiv \rho \Gamma_{(1)}(\rho, V, \Theta) = -\kappa \frac{\partial \Theta}{\partial r} , \tag{100}$$

where

$$\kappa := 3 \rho \tau_* \Theta \tag{101}$$

is called a kinetic coefficient. Therefore, the macroscopic traffic equations (40), (41), and (46) assume the forms

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho V)}{\partial r} = 0 , \tag{102}$$

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial r} = -\frac{1}{\rho} \frac{\partial \mathcal{P}}{\partial r} + \frac{1}{\tau} [V_e(\rho, V, \Theta) - V] , \tag{103}$$

and

$$\frac{\partial \Theta}{\partial t} + V \frac{\partial \Theta}{\partial r} = -\frac{2\mathcal{P}}{\rho} \frac{\partial V}{\partial r} + \frac{1}{\rho} \frac{\partial}{\partial r} \left(\kappa \frac{\partial \Theta}{\partial r} \right) + \frac{2}{\tau} (\mathcal{C} - \Theta) + (1-p) \kappa \frac{\partial \Theta}{\partial r} . \tag{104}$$

Additionally, the corrected *covariance equation* becomes

$$\frac{\partial \mathcal{C}}{\partial t} + V \frac{\partial \mathcal{C}}{\partial r} = -\mathcal{C} \frac{\partial V}{\partial r} - \Theta \frac{\partial V_0}{\partial r} + \frac{1}{\rho} \frac{\partial}{\partial r} \left(\zeta \frac{\partial \Theta}{\partial r} \right) + \frac{1}{\tau} [\mathcal{C}_e(\rho, V, \Theta, \mathcal{C}) - \mathcal{C}] + \frac{(1-p)}{2} \zeta \frac{\partial \Theta}{\partial r} \tag{105}$$

with the kinetic coefficient

$$\zeta := \kappa \frac{\mathcal{C}}{\Theta} = 3 \rho \tau_* \mathcal{C} \tag{106}$$

and the *equilibrium covariance*

$$\mathcal{C}_e(\rho, V, \Theta, \mathcal{C}) := \Theta_0 - 2\tau(1-p) \rho \mathcal{C} \sqrt{\frac{\Theta}{\pi}} . \tag{107}$$

(For a detailed derivation of (105)–(107) cf. Ref. [79].)

Equations (102), (103), and (104) are the *Navier-Stokes-like traffic equations* [58]. Compared with the Navier-Stokes equations for ordinary fluids they possess the additional terms $(V_e - V)/\tau$ and $2(\Theta_e - \Theta)/\tau$ with $\Theta_e = \mathcal{C} + (\tau/2)(1-p)\kappa \partial\Theta/\partial r$ which are due to acceleration and interaction processes. Because of the spatial one-dimensionality of the considered traffic equations, the velocity equation (103) does not include a *shear viscosity term* $(1/\rho)\partial/\partial r(\nu_0 \partial V/\partial r)$. The variance equation (104) is related to the equation of heat conduction. However, Θ does not have the interpretation of "heat" but only of velocity variance here. Finally, the Navier-Stokes-like traffic equations are complemented by the additional covariance equation (105) arising from the tendency of drivers to get ahead with a certain desired velocity v_0.

We recognize that the first-order macroscopic traffic equations (102), (103), (104), and (105) build a closed system of equations. Moreover, according to (98), the relations for the spatial density, average velocity, velocity variance, and traffic pressure did not change. In this sense, the chosen Chapman-Enskog method for closing the hierarchy of macroscopic equations is consistent with its assumption, according to which only the expressions for the flux density of the velocity variance $\mathcal{J} \equiv \rho\Gamma$ and the covariance $\mathcal{C}$ were to be improved by the nonequilibrium correction $\tilde{\rho}_{(1)}(r,v,t)$. However, note that another relation for $\tilde{V}_0(v)$ than (68) would have led to modifications of ρ, V, and/or Θ.

We also recognize that the finite adaptation time τ_* for approaching the equilibrium distribution (66) causes a finite *skewness*

$$\gamma := \frac{\Gamma}{\Theta^{3/2}} = \frac{\mathcal{J}}{\rho\Theta^{3/2}} = -\frac{\kappa}{\rho\Theta^{3/2}} \frac{\partial \Theta}{\partial r} = -\frac{3\tau_*}{\sqrt{\Theta}} \frac{\partial \Theta}{\partial r} \tag{108}$$

of the nonequilibrium velocity distribution

$$P(v; r, t) \approx \frac{\tilde{\rho}_{(0)}(r,v,t) + \tilde{\rho}_{(1)}(r,v,t)}{\rho(r,t)} . \tag{109}$$

This leads to the so-called *transport terms*

$$-\kappa \frac{\partial \Theta}{\partial r} \quad \text{and} \quad -\zeta \frac{\partial \Theta}{\partial r} . \tag{110}$$

The effect of these terms in Eqs. (104) and (105) is to smooth out sudden changes of variance and covariance via the second spatial derivatives of $\Theta(r,t)$, namely,

$$\frac{\partial}{\partial r} \left(\kappa \frac{\partial \Theta}{\partial r} \right) \quad \text{and} \quad \frac{\partial}{\partial r} \left(\zeta \frac{\partial \Theta}{\partial r} \right) . \tag{111}$$

VII. CORRECTIONS OF THE MODEL

A. Driver behavior and bulk viscosity

We remember that the term $-(1/\rho)\partial\mathcal{P}/\partial r$ describes an anticipation effect. It reflects that drivers accelerate when the traffic pressure $\mathcal{P} = \rho\Theta$ lessens, i.e., when the density ρ or the variance Θ decreases. However, drivers additionally react to a spatial change of average velocity. This effect can be modeled by the modified pressure relation

$$\mathcal{P}(\rho, V, \Theta) := \rho\Theta - \eta\frac{\partial V}{\partial r} \tag{112}$$

which gives velocity equation (103) a similar form to the variance equation (104) and covariance equation (105).

In order to present reasons for relation (112) let us assume that drivers switch between two driving modes $m \in \{1, 2\}$ depending on the traffic situation. Let $m = 1$ characterize a *brisk* and $m = 2$ describe a *careful* driving mode. Then we can split the density $\rho(r, t)$ into partial densities $\rho_m(r, t)$ that delineate drivers who are in state m:

$$\rho_1(r, t) + \rho_2(r, t) = \rho(r, t) \,. \tag{113}$$

Both densities are governed by a continuity equation, but this time we have transitions between the two driving modes with a rate $R(\rho_1, V)$ so that

$$\frac{\partial \rho_1}{\partial t} = -\frac{\partial}{\partial r}(\rho_1 V) - R(\rho_1, V) \,, \tag{114a}$$

$$\frac{\partial \rho_2}{\partial t} = -\frac{\partial}{\partial r}(\rho_2 V) + R(\rho - \rho_2, V) \,. \tag{114b}$$

Adding both equations we see that the original continuity equation (102) is still valid. Now, defining the substantial time derivative

$$\frac{D}{Dt} := \frac{\partial}{\partial r} + V\frac{\partial}{\partial r} \tag{115}$$

we can rewrite (114a) and obtain

$$\frac{D\rho_1}{Dt} = -\rho_1\frac{\partial V}{\partial r} - R(\rho_1, V) \,. \tag{116}$$

D/Dt describes temporal changes in a coordinate system that moves with velocity V. Assuming that ρ_1 relaxes rapidly we can apply the adiabatic approximation [78]

$$\frac{D\rho_1}{Dt} \approx 0 \tag{117}$$

which is valid on the slow time scale of the macroscopic changes of traffic flow. This leads to

$$R(\rho_1, V) \approx -\rho_1\frac{\partial V}{\partial r} \,. \tag{118}$$

Relation (117) implies that the density ρ_1 of briskly behaving drivers is approximately constant in the moving coordinate system whereas the density $\rho_2 = \rho - \rho_1$ of carefully behaving drivers varies with the traffic situation:

$$\frac{D\rho_2}{Dt} \approx -\rho\frac{\partial V}{\partial r} \,. \tag{119}$$

ρ_2 increases when the average velocity spatially decreases ($\partial V/\partial r < 0$) since this may indicate a critical traffic situation.

According to relations (114) and (118) incessant transitions between the two driving modes take place as long as traffic flow is spatially nonhomogeneous (i.e., $\partial V/\partial r \neq 0$). This leads to corrections of the pressure relation. Expanding $\mathcal{P}$ with respect to the variable R which characterizes the disequilibrium between the two driving modes we find [74]

$$\mathcal{P}(\rho, \Theta, R) = \mathcal{P}(\rho, \Theta, 0) - \left.\frac{\partial \mathcal{P}}{\partial R}\right|_{R=0} \rho_1\frac{\partial V}{\partial r} + \cdots \,. \tag{120}$$

With the equilibrium relation $\mathcal{P}(\rho, \Theta, 0) = \rho\Theta$ and

$$\eta := \rho_1 \left.\frac{\partial \mathcal{P}}{\partial R}\right|_{R=0} \tag{121}$$

we finally obtain the desired result

$$\mathcal{P}(\rho, \Theta, R) \equiv \mathcal{P}(\rho, V, \Theta) = \rho\Theta - \eta\frac{\partial V}{\partial r} \,. \tag{122}$$

A more detailed discussion can be found in Ref. [74].

B. Modifications due to finite space requirements

We will now introduce some corrections that are due to the fact that vehicles are not pointlike objects but need, on average, a space of

$$s(V) = l + VT \tag{123}$$

each. Here, $l \geq l_0$ is about the *average vehicle length* whereas VT corresponds to the *safe distance* each driver should keep to the next vehicle ahead. T is about the *reaction time.* Consequently, if $\Delta N(r, t) := \rho(r, t)\,\Delta r$ means the number of vehicles that are at a place between $r - \Delta r/2$ and $r + \Delta r/2$, the *effective density* is

$$\varrho(r, t) = \frac{\Delta N(r, t)}{\Delta r - \Delta N(r, t)s[V(r, t)]} = \frac{\rho(r, t)}{1 - \rho(r, t)s[V(r, t)]} \,. \tag{124}$$

Since $\Delta N(r, t)s(V)$ is the space which is occupied by $\Delta N(r, t)$ vehicles, the effective density is the number $\Delta N(r, t)$ of vehicles per effective free space $\Delta r - \Delta N(r, t)s(V)$.

The reduction of available space by the vehicles leads to an increase of their interaction rate. Therefore we have

$$\left(\frac{\partial \hat{\rho}}{\partial t}\right)_{\text{tr}} := (1 - p)\int_v^\infty dw \int dw_0\, |v - w|\hat{\varrho}(r, v, w_0, t)\hat{\rho}(r, w, v_0, t) - (1 - p)\int_0^v dw \int dw_0\, |w - v|\hat{\varrho}(r, w, w_0, t)\hat{\rho}(r, v, v_0, t) \tag{125}$$

with

$$\hat{\varrho}(r,v,v_0,t) := \frac{\hat{\rho}(r,v,v_0,t)}{1-\rho(r,t)s[V(r,t)]} \,. \tag{126}$$

Consequently, we obtain the corrected relation

$$\frac{1}{\tau_0} := (1-p)\varrho\sqrt{\frac{\Theta}{\pi}} \,. \tag{127}$$

In addition, we must replace $\mathcal{P}$ and $\mathcal{J}$ by

$$\mathcal{P}' := \frac{\mathcal{P}}{1-\rho s(V)} \quad \text{and} \quad \mathcal{J}' := \frac{\mathcal{J}}{1-\rho s(V)}, \tag{128}$$

respectively [76]. For the kinetic coefficients η, κ, and ζ we obtain the corrected relations

$$\eta' := \frac{\eta}{1-\rho s(V)} \,, \qquad \kappa' := \frac{\kappa}{1-\rho s(V)} = 3\varrho\tau_*\Theta \,,$$

$$\text{and} \qquad \zeta' := \frac{\zeta}{1-\rho s(V)} = 3\varrho\tau_*\mathcal{C} \,. \tag{129}$$

The corrected formula

$$\varrho\Theta = \frac{\rho\Theta}{1-\rho s(V)} \tag{130}$$

for the equilibrium pressure corresponds to the pressure relation of van der Waals for a "real gas." According to (130), the traffic pressure diverges for $\rho \to \rho_{\max} := 1/l$ which causes a deceleration of vehicles.

The corrected kinetic coefficients $\eta'(\rho,V,\Theta)$, $\kappa'(\rho,V,\Theta)$, and $\zeta'(\rho,V,\Theta,\mathcal{C})$ also diverge for $\rho \to \rho_{\max}$ [76]. We find, for example,

$$\kappa' \stackrel{\rho\approx\rho_{\max}}{\longrightarrow} 3\varrho\tau'\Theta = \frac{3\rho\tau'\Theta}{1-\rho s(V)} \tag{131}$$

so that the divergence of κ' is a consequence of the finite reaction and braking time τ'. This divergence causes a homogenization of traffic flow since the second spatial derivatives $\partial/\partial r(\eta\partial V/\partial r)$, $\partial/\partial r(\kappa\partial\Theta/\partial r)$, and $\partial/\partial r(\zeta\partial\Theta/\partial r)$ produce a spatial smoothing of the average velocity V, variance Θ, and covariance $\mathcal{C}$, respectively.

It is the divergence of traffic pressure and kinetic coefficients for $\rho \to \rho_{\max}$ that prevents the spatial density ρ from exceeding the maximum density $\rho_{\max}$ [5].

VIII. SUMMARY AND OUTLOOK

This paper started with a discussion of the most widespread macroscopic traffic models. Each of them is suitable for the description of certain traffic situations on freeways but fails for others. Therefore, an improved fluid-dynamic model was derived from the gas-kinetic traffic equation of Paveri-Fontana [2] which is very well justified and does not show the peculiar properties of Prigogine's Boltzmann-like approach [4].

For the derivation of the improved traffic model, moment equations for collective (macroscopic) quantities like the spatial density, average velocity, and velocity variance had to be calculated. The system of macroscopic equations turned out to be nonclosed so that a suitable approximation was necessary. Here, the well proved Chapman-Enskog method was applied. In zeroth-order approximation the velocity distribution is assumed to be in local equilibrium. According to empirical data, the latter is characterized by a Gaussian velocity distribution. Depending on the respective kind of zeroth-order approximation one arrives at the Lighthill-Whitham model [10], the model of Phillips [3,71], or the Euler-like traffic equations.

For the derivation of a first-order approximation, the reduced Paveri-Fontana equation was linearized around the local equilibrium solution and solved by application of the Euler-like traffic equations. The resulting correction term for the nonequilibrium velocity distribution allowed the calculation of additional transport terms which describe the flux density of velocity variance and covariance in spatially nonhomogeneous situations. They are related to a finite skewness of the velocity distribution. The shear-viscosity term vanishes because of the one-dimensionality of the considered traffic equations. Nevertheless, a bulk-viscosity term results from transitions between two different driving modes: a brisk and a careful one.

The resulting Navier-Stokes-like traffic equations were finally corrected in order to take into account the finite space requirements of vehicles. They overcome the deficiencies of the former macroscopic traffic models so that the criticism by Daganzo [51] and others could be invalidated.

(1) The anticipation term which, in other models, is responsible for the prediction of negative velocities vanishes in problematic situations like the one described at the end of Sec. II since the variance becomes zero then.

(2) The density $\rho(r,t)$ does not exceed the maximum admissible density ρ_{bb} (= bumper-to-bumper density) [5] since the diverging viscosity term causes a homogenization of traffic flow and the diverging traffic pressure suppresses an unrealistic growth of velocity which stops further increase of traffic density.

(3) The model takes into account different driving styles by a distribution of desired velocities v_0 which are directly associated with the individual drivers. An extension of the Navier-Stokes-like traffic equations to different vehicle types (cars and trucks) is possible [88].

(4) The interaction between drivers is modeled anisotropically since the slower vehicle is assumed not to be affected by a faster vehicle behind it or overtaking it.

(5) According to the Navier-Stokes-like equations, disturbances may propagate with a velocity $c > V$ since a certain proportion of vehicles moves faster than the average velocity V due to the finite velocity variance Θ. Therefore, in contrast to what was claimed by Daganzo [51], it is admissible that macroscopic traffic models "exhibit one characteristic speed greater than the macroscopic fluid velocity" [51,89].

Present investigations focus on the computer simulation of the Navier-Stokes-like traffic equations. This work has already been successfully started for a circular road [5,90] and is now extended to complex freeway networks.

Moreover, the gas-kinetic and Navier-Stokes-like traffic models can be generalized to models for multilane traffic where overtaking and lane changing are explicitly taken into account [88]. By this method, formulas for the relations $\tau(\rho, V, \Theta)$ and $p(\rho, V, \Theta)$ can be derived [91].

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[67] This derivation is not really consistent since the *relaxation time* $\tau \approx 10$ s in (8) is not of the order of magnitude which τ has in the follow-the-leader models where it has the meaning of a *reaction time* (≈ 1.3 s [41,23]) [45]. Another criticism was pointed out by Daganzo [51].

[68] Payne's simulation model FREFLO, in contrast to his macroscopic model (1) and (8), predicts unstable traffic flow above a critical density that depends on the spatial discretization Δr [45].

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[77] Probably Paveri-Fontana's equation (27) should, on its right-hand side, be extended by the additional *adaptation term* $(\partial\hat{\rho}/\partial t)_{\rm ad} := \hat{\rho}(r,v,t)[P_0'(v_0;r,t) - P_0(v_0;r,t)]/T$, where $T \approx 1$ s is about the reaction time. This term is intended to describe an adaptation of the actual distribution of desired velocities $P_0(v_0;r,t)$ to the reasonable distribution of desired velocities $P_0'(v_0;r,t) \approx \exp\{-[v_0 - V_0'(r,t)]^2/[2\Theta_0'(r,t)]\}/\sqrt{2\pi\Theta_0'(r,t)}$. The mean value $V_0'(r,t)$ and variance Θ_0' of $P_0'(v_0;r,t)$ mainly depend on speed limits and road conditions (gradient of the road; fog, rain, snow, or ice on the road). Therefore, they are constant almost everywhere. Applying the method of adiabatic elimination [78], which is applicable due to the smallness of T, we find $P_0(v_0;r,t) \approx P_0'(v_0;r,t)$. This implies that the average desired velocity $V_0(r,t) := \int dv \int dv_0\, v_0\hat{\rho}(r,v,v_0,t)/\rho(r,t)$ and the variance of desired velocities $\Theta_0(r,t) := \int dv \int dv_0\, (v_0 - V_0)^2\hat{\rho}(r,v,v_0,t)/\rho(r,t)$ are given by the external traffic conditions that determine $V_0'(r,t)$ and $\Theta_0'(r,t)$: $V_0(r,t) \approx V_0'(r,t)$ and $\Theta_0(r,t) \approx \Theta_0'(r,t)$. Interestingly enough, the adaptation term does not bring about modifications of the density equation, the velocity equation, the variance equation, or the covariance equation derived later on. Therefore it was not included in the main discussion.

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[87] To illustrate this, assume that we are confronted with a linear differential equation $d\vec{x}/dt = -\underline{L}\vec{x}$ with a linear operator (or matrix) $\underline{L}$. This equation has the general solution $\vec{x}(t) = \sum_\mu a_\mu \vec{x}_\mu e^{-t/\tau_\mu}$ where the eigenvalues $1/\tau_\mu$ are the solutions of the characteristic equation $\det(\underline{L} - 1/\tau_\mu) = 0$ and the eigenvectors $\vec{x}_\mu$ satisfy the equation $\underline{L}\vec{x}_\mu = \vec{x}_\mu/\tau_\mu$. If $1/\tau_\mu \gg 1/\tau_0$ for $\mu \neq 0$ we have $\vec{x}(t) \approx a_0\vec{x}_0 e^{-t/\tau_0}$ after a very short time $t > 3\max\{\tau_\mu : \mu \neq 0\}$ (adiabatic approximation). This implies $-\underline{L}\vec{x}(t) = d\vec{x}(t)/dt \approx -\vec{x}(t)/\tau_0$.

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[89] Here, the term "characteristic speeds" refers to the slopes of the characteristics in the (r,t) plane which can be calculated for a hyperbolic system of partial differential equations [63].

[90] However, note that the traffic model simulated in Ref. [5] differs in some details from the Navier-Stokes-like model derived in this paper.

[91] If one assumes $\tau(\rho,V,\Theta) \equiv \tau(\rho)$ and $p(\rho,V,\Theta) \equiv p(\rho)$ as Prigogine and Herman [4], Phillips [3], and Paveri-Fontana [2] did, the unknown functional relation for the product $\tau(\rho)[1-p(\rho)]$ can be determined from the empirical velocity-density relation $V_e(\rho_e)$ by inserting (82) into (80).